Finite Element Methods Used in the Axisymmetric Linear FE Module

This module implements a small-strain, axisymmetric finite element formulation designed to analyze bonded rubber cylinders and cones subjected to compression. It supports various geometries (cylinder, straight cone, and curved cone) and near-incompressible materials defined by shear modulus G and bulk modulus K.

1. Element Type and Geometry

The model uses 4-node bilinear quadrilateral (Q4) axisymmetric elements. Each element represents a ring section in the r–z plane and assumes circular symmetry about the axis (r=0). The mesh is generated from user-defined outer and inner diameters that may vary linearly or with curvature defined by quadratic parameters kout and kin.

2. Material Model

The rubber material is modeled as linear isotropic elastic in terms of shear modulus G and bulk modulus K. The Lamé constants are derived as $\mu=G$ and $\lambda=K-2G/3$. Poisson's ratio is computed as $\nu=(3K-2G)/(2(3K+G))$ for reference.

3. Selective Reduced Integration

To mitigate volumetric locking, selective reduced integration is used:
- The deviatoric (shear) part of the stiffness matrix is integrated using 2×2 Gauss points with $\lambda=0$.
- The volumetric (bulk) part is integrated using a single Gauss point at the element center.

This ensures stable results even for nearly incompressible materials where $\nu\approx0.5$.

4. Axisymmetric Strain–Displacement Matrix (B)

The strain components are defined as:
$\varepsilon_r = \partial u_r/\partial r$, $\varepsilon_z = \partial u_z/\partial z$, $\varepsilon_\theta = u_r/r$, and $\gamma_{rz} = \partial u_r/\partial z + \partial u_z/\partial r$.
These are assembled into a 4×8 matrix B for each element, mapping nodal displacements to strains.

5. Global Assembly

Each element's stiffness contribution Ke is computed as $Ke = \int B^T D B \, 2\pi r \det(J) \, d\xi d\eta$, where D is the constitutive matrix (either deviatoric or volumetric). The global stiffness matrix K is assembled by summing element contributions.

6. Boundary Conditions

Boundary conditions applied:
- ur=0 along the axis of symmetry (r=0).
- uz=0 at the bottom surface (z=0).
- uz=−Δ on the top surface for imposed displacement.
- Optional ur=0 at top and bottom surfaces for fully bonded ends.

The user can toggle the radial constraint mode in the UI.

7. Solution Method

The reduced linear system KUuU = −KUKuK is solved using Cholesky decomposition, which is efficient for symmetric positive-definite matrices. Small diagonal regularization is added to prevent numerical instability.

8. Reaction Forces and Stiffness

After solving for displacements, reaction forces $R = Ku$ are computed. The total compressive reaction on the top surface is summed to yield the global load F. The effective stiffness is then $k = F/\Delta$.

9. Post-Processing and Visualization

• The undeformed and deformed meshes are plotted together for visual comparison.
• Shear strain γ_{rz} is evaluated at multiple Gauss points per element and displayed as color-filled contours from blue (min) to red (max).
• The legend updates automatically to show min and max γ_{rz} values.

10. Numerical Stability and Notes

The method maintains numerical stability for nearly incompressible materials by using selective reduced integration and regularized Cholesky solving. Mesh density should be increased ($nr \approx 32$, $nz \approx 12$) for smoother contours and more accurate stress gradients.