Vertical Plate — Steady-State Natural Convection Heat Sink

This module provides a quick, engineering-level estimate of the steady-state temperature field in a thin vertical plate cooled by one-sided natural convection. Users define the plate geometry, material, and ambient air temperature, then draw rectangular heat sources on the plate surface. The code assembles a finite-volume conduction model in the plate with lumped heat sources and a convective sink term based on the Churchill–Chu natural convection correlation for a vertical plate. The solution is displayed as a color contour map of plate temperature, along with summary metrics such as average temperature, maximum temperature, approximate convection coefficient, and an energy-balance residual.

1. Physical Model and Assumptions

- Geometry: uniform vertical plate of width W (m), height H (m), and thickness t (m). The plate is discretized into a uniform Cartesian grid of $n \times m$ finite volume cells.
- Materials: the code includes several common materials (Aluminum 6061, Copper, Steel, FR4) with constant thermal conductivity k, density ρ, and specific heat cp. Only k and t appear in the steady-state conduction equations; ρ and cp are reserved for potential transient extensions.
- Heating: each rectangular region drawn on the canvas is treated as a uniform heat source with specified total power W (Watts). The rectangle is mapped to the underlying grid cells, and the total power is divided equally among those cells to obtain a per cell heat generation rate Qi (W).
- Convection: the plate exchanges heat with still air at an ambient temperature Ta (°C) on one side only. The local convective heat flux is modeled as q_conv = h (T - T_a), where h ($W/m^2 \cdot K$) is an effective, plate-averaged natural convection coefficient computed from the Churchill–Chu correlation for a vertical plate.
- Air properties: the Churchill–Chu correlation is evaluated using fixed air properties representative of near-room conditions: kair = 0.026 W/m·K, $\nu = 1.5\times10^{-5}$ m^2/s, $\alpha = 2.2\times10^{-5}$ m^2/s, Pr ≈ 0.71, $\beta \approx 1/300$ K^{-1}, and g = 9.81 m/s^2.
- Steady state: the solution is purely steady-state. Transient behavior and thermal capacity are not modeled; the code iterates directly to the steady temperature field for a given set of heat sources.
- Two-dimensional conduction: conduction is modeled in the plate plane (width–height). Through-thickness conduction is collapsed into an effective conductance using k and t; temperature is assumed uniform through the thickness at each (x, y) location.
- One-sided convection: all convective heat loss is applied to the plate surface facing the air. The back side is assumed either adiabatic or otherwise not participating in convection; this is built into the energy balance via a per-cell sink term proportional to h.

- Laminar natural convection regime: the Churchill–Chu correlation used is valid for laminar free convection on a vertical plate over a broad range of Rayleigh numbers. The module implicitly assumes that the flow remains within the correlation's validity range.

2. Natural Convection Correlation (Churchill–Chu)

The code uses a standard Churchill–Chu correlation for laminar and transitional natural convection over a vertical plate to estimate the average Nusselt number Nu_L over the plate height L. In simplified form, the correlation is:

$$Ra_L = g\ \beta\ (\Delta T)\ L^3 / (\nu\ \alpha)$$
$$Nu_L = 0.68 + [0.670\ Ra_L^{1/4}] / [\ (1 + (0.492/Pr)^{9/16})^{4/9}\ ]$$
$$h = Nu_L\ k_{air} / L$$

Here RaL is the Rayleigh number based on plate height L and characteristic temperature difference ΔT, Pr is the Prandtl number, and kair is the thermal conductivity of air. In the implementation:

- L is taken as the full plate height H.
- ΔT is taken as max($\bar{T}$ - Ta, 0), where $\bar{T}$ is the average plate temperature over all cells and Ta is ambient air temperature.
- Air properties (kair, ν, α, Pr, β) are held constant at near-room values for simplicity.

3. Finite-Volume Discretization of the Plate

The conduction problem in the plate is solved using a two-dimensional finite-volume (FV) method on a rectangular, uniform grid. Let i and j index the cell centers in the x (width) and y (height) directions. Each cell has dimensions Δx and Δy, area A = Δx Δy, and plate thickness t.

For an internal cell (i, j), energy conservation at steady state (per unit depth t) can be written as:

$$(k\ t / \Delta x^2)\ (TW - T_P) + (k\ t / \Delta x^2)\ (TE - TP)$$
$$+ (k\ t / \Delta y^2)\ (TS - T_P) + (k\ t / \Delta y^2)\ (TN - TP)$$
$$+ q_{conv} + q_{gen} = 0$$

where T_P is the unknown temperature in the current cell, TW, TE, TS, TN are temperatures in the west, east, south, and north neighbors, q_conv is the convective loss to the surrounding air, and q_gen is the volumetric heat generation from sources mapped to the cell.

The convective term is modeled as a lumped sink term per unit planform area:

$$qconv = h\ (TP - Ta)$$

The total heat generation in the cell is Qi (W) from the drawn sources; on a per-area basis this becomes:

$qgen = Qi / A$

Grouping terms gives a standard five-point stencil form:

$$aP\ TP = aW\ TW + aE\ TE + aS\ TS + aN\ TN + b$$

with coefficients:

$$aW = aE = k\ t / \Delta x^2$$
$$aS = aN = k\ t / \Delta y^2$$
$$aP = aW + aE + aS + aN + h$$
$$b = h\ Ta + Q_i / A$$

Boundary cells are treated with a zero-normal-flux approximation by reusing the cell's own temperature for missing neighbors (e.g., TW = TP at the left boundary), while still applying the convective sink term uniformly. This produces a simple and robust discrete system suitable for rapid engineering estimates.

4. Solution Algorithm (SOR + Picard Iteration on h)

The discretized system is nonlinear because the convective coefficient h depends on the plate-average temperature via the Churchill–Chu correlation. The code handles this nonlinearity using a simple outer Picard iteration on h combined with an inner Successive Over-Relaxation (SOR) loop for the linear system at fixed h.

The overall algorithm can be summarized as follows:

1. Initialize: build the grid based on user-specified width, height, number of cells along the width (nx), and material; compute ny from the aspect ratio. Set all cell temperatures to ambient Ta.
2. Map heat sources: for each user-defined rectangle, map its extents to the current grid to identify the covered cell indices; distribute the total source power W evenly across those cells to obtain a per-cell source Qi.
3. Outer loop on h (Picard iteration):
   a. With the current estimate of h, form the FV coefficients aP, aW, aE, aS, aN, and right-hand side b.
   b. Solve the linear system using pointwise SOR. At each sweep, update each cell temperature as:
   $$Tnew = (aW\ TW + aE\ TE + aS\ TS + aN\ TN + b) / aP$$
   $$T \leftarrow T + \omega\ (Tnew - T)$$
   where ω is the over-relaxation factor (typically $\approx$ 1.8–1.9). Iterate until the maximum change in any cell $\Delta Tmax$ falls below a specified tolerance.
4. Update h: once the inner SOR loop converges, compute the average plate temperature $\bar{T}$ over all cells and re-evaluate the Churchill–Chu correlation using $\Delta T = \max(\bar{T} - T_a, 0)$. This yields a new h_{new}. For robustness, the code under-relaxes the update:
   $$h \leftarrow 0.5\ h + 0.5\ h_{new}$$

5. Convergence check on h: if the relative change |hnew - hold| / max(hold, ε) is smaller than a small tolerance (e.g., 1%), the outer loop terminates. Otherwise, rebuild the coefficients with the updated h and repeat the inner SOR solve.

In addition to convergence on h and the SOR residual, the code computes an overall energy-balance residual:

residual = |Qin - Qout| / (|Qin| + |Qout|)
Qin = Σ W_s (sum of user specified heat source powers)
Qout = Σ_cells [h Acell max(Ti - Ta, 0)]

This residual is reported as a percentage in the topline summary and provides a quick check that the steady solution approximately conserves energy (heat in ≈ heat out).

5. Using the Module

6. Specify plate geometry and material: enter width W, height H, and thickness t in meters, and select the material (Aluminum 6061, Copper, Steel, or FR4). Choose the number of cells along the width (nx). The code automatically computes ny based on the aspect ratio H/W.
7. Set ambient conditions: enter the ambient air temperature Ta (°C). This serves as the reference for the natural convection correlation and the convective sink term.
8. Setup grid: click "Setup Grid" to build the finite-volume mesh and reset the temperature field to ambient. Optionally enable or disable the grid overlay for visualization.
9. Define heat sources: specify a default power level (W) for new sources. Then, click and drag rectangles on the canvas to create heater regions. Each rectangular source appears in the sources list, where its power can be edited and the source can be removed. The "Clear all sources" button removes all sources at once.
10. Solve: click "Solve" to run the SOR + Picard iteration sequence. The top status line reports the grid size, current h, plate area, total applied power ΣW, energy residual, average plate temperature, and maximum cell temperature. The canvas shows a color temperature map with a legend for Tmin and Tmax.
11. Reset: click "Reset to Ambient" to set the entire plate back to ambient temperature while retaining the current geometry, material, and sources. This is useful when exploring sensitivity to parameters such as ambient temperature or plate thickness.

6. Suggested Parameter Ranges and Usage Notes

- Grid resolution: for smooth temperature fields, nx in the range 60–120 is usually sufficient. Very fine grids (nx > 200) increase computational cost without substantial accuracy gains for this level of modeling.
- Plate dimensions: the Churchill–Chu correlation assumes a "large" vertical surface relative to the boundary layer thickness. Plates from a few centimeters to several tens of centimeters in height with moderate temperature differences are typical for natural convection electronics cooling problems.

- Heat source powers: the module is linear in total power for a fixed h, but because h is updated based on ΔT, very high total powers may push the system into a regime where the laminar correlation is less accurate. For large ΔT (e.g., > 50–60 K), users should treat results as approximate trend indicators.
- Material selection: high-conductivity materials such as aluminum and copper reduce temperature gradients within the plate, while lower-conductivity materials (FR4, steel) produce more localized hot spots for a given source distribution.

7. Limitations of the Model

- One-sided lumped convection: the entire plate is cooled using a single averaged convection coefficient h based on the plate-average temperature. Local variations in heat transfer coefficient along the plate height or width are not resolved.
- No radiation: thermal radiation to the surroundings is neglected. For high-temperature applications or low-convection environments, radiative losses may be comparable to or larger than natural convection.
- No fluid domain solution: the surrounding air flow is not solved explicitly; the flow behavior is entirely embedded in the Churchill–Chu correlation. Complex geometries, enclosures, or neighboring hot surfaces are not modeled.
- Laminar correlation only: the Churchill–Chu correlation used is most accurate for laminar and transitional natural convection. Strong heating that drives the Rayleigh number far into the turbulent regime is not explicitly handled.
- Uniform through-thickness temperature: the plate is treated as thermally thin. Temperature variations across the thickness are ignored; this is typically reasonable when t is small compared to characteristic in-plane dimensions and k is not extremely low.
- Simplified boundary conditions: edges are treated with a simple zero-normal-flux approximation plus the uniform convective sink. Fine details such as edge fins, cutouts, or mounting hardware are not represented.

8. References

1. S. W. Churchill and H. H. S. Chu, "Correlating Equations for Laminar and Turbulent Free Convection from a Vertical Plate," International Journal of Heat and Mass Transfer, Vol. 18, pp. 1323–1329, 1975.

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3. S. V. Patankar, Numerical Heat Transfer and Fluid Flow, Hemisphere, 1980. (Classical reference on finite-volume discretization and SOR solution strategies.)

4. A. Bejan, Convection Heat Transfer, 4th ed., Wiley, 2013. (Background on natural convection correlations and scaling for vertical plates.)